Chromatic Polynomials of Some Mixed Hypergraphs

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Abstract

Motivated by a recent result of Walter [19] concerning the chromatic polynomials of some hypergraphs, we present the chromatic polynomials of several (non-uniform) mixed hypergraphs. We use a recursive process for generating explicit formulae for linear mixed hypercacti and multi-bridge mixed hypergraphs using a decomposition of the underlying hypergraph into blocks, defined via chains. Further, using an algebra software package such as Maple, one can use the basic formulae and process demonstrated in this paper to generate the chromatic polynomials for any linear mixed hypercycle, unicyclic mixed hypercyle, mixed hypercactus and multi-bridge mixed hypergraph. We also give the chromatic polynomials of several examples in illustration of the process including the formulae for some mixed sunflowers.

Keywords: Hypergraph coloring, cohypergraph, bihypergraph, mixed hypergraph, hypercycle, hypercactus, multi-bridge hypergraph, $\Theta-$hypergraph, sunflower, chromatic polynomial.

1   Introduction

A *hypergraph* $\mathcal{H}$ is an ordered pair $(X, \mathcal{E})$ where X is a finite set of vertices, with *order* $|X| = n$, and $\mathcal{E}$ is a collection of nonempty subsets of X. The number of elements of X contained in an element e of $\mathcal{E}$, denoted $|e|$, is the *size* of e. When $|e| = 2$ for all elements of $\mathcal{E}$, then $\mathcal{H}$ is a *graph* and the elements of $\mathcal{E}$ are its *edges*. More generally, $\mathcal{H}$ is a hypergraph and the elements of $\mathcal{E}$ are its *hyperedges*. The *degree* of a vertex v, denoted $d(v) = d_{\mathcal{H}}(v)$, is the number of hyperedges of $\mathcal{H}$ containing v. $\mathcal{H}$ is *Sperner* if no hyperedge is a subset of another hyperedge. $\mathcal{H}$ is said to be *linear* if $e_1 \cap e_2$ is either empty or a singleton for any pair of hyperedges. If all hyperedges of $\mathcal{H}$ have size k, then we say $\mathcal{H}$ is $k-uniform$. Obviously a $2-$uniform hypergraph is a graph. For further basic definitions of graphs and hypergraphs, we refer the reader to [5, 20].

Much progress has been made to extend and generalize several theories of graphs to hypergraphs. In particular, vertex coloring is an active area of ongoing research [6, 7, 9, 13, 14, 15, 16]. A $\lambda-coloring$ of a hypergraph $\mathcal{H}$ is a mapping $f : X \to \{1, 2, \dots, \lambda\}$. A surjective

mapping f is a *strict* coloring. In graph theory, a coloring is a *proper* coloring if $f(u) \neq f(v)$ any time u and v are adjacent, meaning $\{u, v\} \in \mathcal{E}$. The natural extension to hypergraphs, wherein f is a proper coloring if at least two vertices of each hyperedge are assigned different colors, has been studied extensively and is usually what is meant by hypergraph coloring. However, some authors have studied the inverse condition, wherein at least two vertices of each hyperedge are given the same color. Others have even required the combination of both conditions simultaneously, which is only possible for (hyper)edges of size at least 3. When hypergraph vertex coloring is studied, we partition $\mathcal{E}$ into three disjoint subsets $\mathcal{E} = \mathcal{C} \cup \mathcal{D} \cup \mathcal{B}$ and denote the hypergraph $\mathcal{H}$ by the four-tuple $(X, \mathcal{C}, \mathcal{D}, \mathcal{B})$. In this context $\mathcal{H}$ is a *mixed* hypergraph. Besides being used to encode partitioning constraints, mixed hypergraphs theory has several other applications, notably in communications models for cyber security [11].

A *proper coloring* of a mixed hypergraph is a $\lambda-$coloring such that f assigns the same color to at least one pair of vertices of each hyperedge in $\mathcal{C}$, f assigns different colors to at least one pair of vertices in each hyperedge in $\mathcal{D}$, and f assigns the same color to one pair and different colors to another pair of each hyperedge of $\mathcal{B}$. Colorings f_1 and f_2 are different if $f_1(u) \neq f_2(u)$ for at least one vertex u. The *chromatic polynomial $P(\mathcal{H}) = P(\mathcal{H}, \lambda)$* counts the number of proper $\lambda-$colorings of $\mathcal{H}$. Note that it is customary (in [18] for instance) to write $\mathcal{E}$ as the union of two not necessarily disjoint sets $\mathcal{C}$ and $\mathcal{D}$ and define $\mathcal{B}$ to be their intersection. For the purposes of writing explicit formulae for chromatic polynomials, it is more convenient to require $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{B}$ to be disjoint.

The subhypergraph $\mathcal{H}[V]$ induced by a set of vertices $V \subset X$ is the hypergraph with vertices V and the hyperedges of $\mathcal{H}$ which are contained in V. The subhypergraph $\mathcal{H}[F]$ induced by a set of hyperedges F is the subhypergraph induced by the vertices of the elements of F. The subhypergraph $\mathcal{H}[\mathcal{D}]$ is a $\mathcal{D}-$hypergraph, or simply a hypergraph. The subhypergraph $\mathcal{H}[\mathcal{C}]$ is a $\mathcal{C}-$hypergraph or *cohypergraph*. The subhypergraph $\mathcal{H}[\mathcal{B}]$ is a $\mathcal{B}-$hypergraph, or *bihypergraph*.

Explicit formulae for the chromatic polynomials for some types of hypergraphs have been given by many authors [6, 7, 8, 19]. Fewer formulae are known for cohypergraphs, bihypergraphs, and general mixed hypergraphs (see for instance [3, 4, 15, 18]), though much has been studied about these colorings and related issues, such as the *feasible* set of integer values of λ for which there is a strict proper coloring [12, 13, 14, 15, 17, 18]. Our contribution is to extend some recent results of Walter concerning some $\mathcal{D}-$hypergraphs to their mixed hypergraphs counterparts, thus, adding to the very few known literature concerning the chromatic polynomials of mixed hypergraphs. To achieve this purpose, this paper builds on some explicit formulae given in [3] to find chromatic polynomials for a larger collection of mixed hypergraphs. In section 2 we review a way to decompose mixed hypergraphs and give a slightly more general version of a well-known splitting theorem for the chromatic polynomials of certain separable mixed hypergraphs. In section 3 we review the results of [3], as well as comment on formulations of some classes of hypergraphs. In section 4 we extend the results of [3] to obtain a process to generate the chromatic polynomial of any linear mixed hypercycle. In section 5 we give a general expression for the chromatic polynomial of any linear mixed hypercactus based on the formulae of sections 3 and 4, and

give some examples to illustrate these formulae. In section 6 we give an expression for the chromatic polynomial of linear multi-bridge mixed hypergraphs which can be referred to as mixed $\Theta-$hypergraphs (or $\Theta-$mixed hypergraphs). To conclude, in section 7, we derive the formulae for sunflower mixed hypergraphs, which are an example of nonlinear mixed hypergraphs. Their blocks which are sunflowers, can be included to increase the class of mixed hypergraphs whose chromatic polynomials can be computed using the splitting theorem of section 2 or a combinatorial argument. We present the latter method.

2   Decompositions of Hypergraphs

The subset $X_0 \subset X$ is a *separator* of the connected hypergraph $\mathcal{H}$ if there are nonempty pairwise disjoint subsets $X_1, \ldots, X_k$, $k \geq 2$, such that $X = \bigcup_{i=0}^{k} X_i$ and no hyperedge containing $x \in X_i$ also contains $y \in X_j$ for any $1 \leq i < j \leq k$. The induced subhypergraphs $\mathcal{H}_i = \mathcal{H}[X_i \cup X_0]$ for $1 \leq i \leq k$ are the *derived* subhypergraphs with respect to X_0, which is the induced subhypergraph $\mathcal{H}_0 = \mathcal{H}[X_0]$.

Voloshin [17] gives a decomposition theorem for chromatic polynomials which generalizes a property of complete subgraphs, or cliques, to hypergraphs. The hypergraphs performing this role are the *uniquely colorable* mixed hypergraphs, which are hypergraphs that have a unique strict proper coloring, up to permutation of the colors.

Theorem 2.1. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D}, \mathcal{B})$ be a connected mixed hypergraph with derived subhypergraphs $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k$ with respect to a uniquely colorable separator $\mathcal{H}_0$. Then*

$$P(\mathcal{H}) = \Big(P(\mathcal{H}_0) \Big)^{1-k} \prod_{i=1}^{k} P(\mathcal{H}_i). \tag{1}$$

It is easy to construct examples where the chromatic polynomial does not decompose in this way for separators that are not uniquely colorable. See Figure 1.

Example 2.1. *Let $\mathcal{H}$ be the $\mathcal{D}-$hypergraph with two hyperedges of size 3 and a common intersection of size 2. (See Figure 1).*
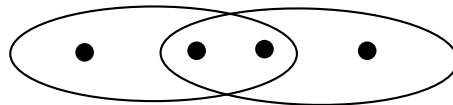

Figure 1: A sunflower with two petals

Then, the derived subhypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ are the hyperedges with respect to the separator $\mathcal{H}_0 \cong \overline{K_2}$ which is the empty graph on two vertices. The chromatic polynomials $P(\mathcal{H}) = \lambda(\lambda - 1)(\lambda^2 + \lambda - 1)$, $P(\mathcal{H}_1) = P(\mathcal{H}_2) = \lambda(\lambda^2 - 1)$, and $P(\mathcal{H}_0) = \lambda^2$ do not satisfy (1).

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However, it is easy to extend Theorem 2.1, recursively, when derived subhypergraphs with respect to one uniquely colorable subhypergraph can be separated again by another uniquely colorable subhypergraph. We state this result in a general form, and then give a simpler form when singletons repeatedly separate the hypergraph which we use in the majority of this paper.

Corollary 2.1.1. *Let X_0^1 separate the connected mixed hypergrpah $\mathcal{H}$ into two derived subhypergraphs $\mathcal{H}_1^1$ and $\mathcal{H}_2^1$ with respect to a uniquely colorable separator $\mathcal{H}_0^1$. Let X_0^i separate $\mathcal{H}_2^{i-1}$ into two derived subhypergraphs $\mathcal{H}_1^i$ and $\mathcal{H}_2^i$ with respect to a uniquely colorable separator $\mathcal{H}_0^i$ for $2 \leq i \leq k$. Then*

$$P(\mathcal{H}) = \frac{P(\mathcal{H}_1^1)P(\mathcal{H}_2^1)}{P(\mathcal{H}_0^1)} = \ldots = \frac{P(\mathcal{H}_1^1)\ldots P(\mathcal{H}_1^k)P(\mathcal{H}_2^k)}{P(\mathcal{H}_0^1)\ldots P(\mathcal{H}_0^k)}. \tag{2}$$

We now define blocks of a hypergraph, following Walter [19] and Acharya [1]. A *chain* in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is an alternative sequence of vertices and hyperedges $v_1, e_1, v_2, e_2, \ldots, e_m, v_{m+1}$ where $v_i \neq v_j$ for $1 \leq i < j \leq m$ and $\{v_i, v_{i+1}\} \subset e_i$. If the hyperedges are also distinct, i.e., $e_i \neq e_j$ for $1 \leq i < j \leq m$, the chain is a *path* of length m. A chain with $v_1 = v_{m+1}$ is a *cyclic chain* and a path with $v_1 = v_{m+1}$ is a *cycle*.

The relation $\sim$ on $\mathcal{E}$ defined by $e_1 \sim e_2$ if and only if $e_1 = e_2$ or there is a cyclic chain containing both e_1 and e_2 is an equivalence relation [1]. A *block* of $\mathcal{H}$ is either an isolated vertex or a subhypergraph induced by the hyperedge set of an equivalence class. This definition is a natural generalization of the definition of blocks for graphs [20], as Acharya shows with

Lemma 2.1. *Two distinct blocks of a hypergraph have at most one vertex in common.*

Though a block of a hypergraph can intersect two or more other blocks nontrivially, there must always be a block which has a single vertex in common with the union of the other blocks in the collection; else we can construct a cyclic chain contradicting the definition of blocks. Since our hypergraphs are finite, we can order the blocks in the following convenient way, which we state as

Corollary 2.1.2. *There is an ordering of the blocks $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k$ of a connected hypergraph $\mathcal{H}$ so that $\bigcup_{i=2}^{k} \mathcal{H}_i \cap \mathcal{H}_1 = \{x_0^1\}$ and $\bigcup_{i=1}^{j-1} \mathcal{H}_i \cap \mathcal{H}_j = \{x_0^j\}$ for $2 \leq j \leq k$.*

Since a single vertex, viewed as the induced subhypergraph $\mathcal{H}_0^i = \mathcal{H}[x_0^i]$, is trivially uniquely colorable, the above ordering of the blocks of a mixed hypergraph provides an iterated set of separators and derived subhypergraphs $\mathcal{H}_i$ as in Corollary 2.1.1 (with appropriate changes to names), and (2) immediately gives the

Corollary 2.1.3. *Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k$ be the blocks of a connected hypergraph $\mathcal{H}$. Then*

$$P(\mathcal{H}) = \lambda^{1-k} \prod_{i=1}^{k} P(\mathcal{H}_i). \tag{3}$$

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Note: Ordering the blocks as in Corollary 2.1.2 makes the proof of Corollary 2.1.3 immediate from Corollary 2.1.1, but expression (3) is symmetric and, therefore, independent of the ordering of the blocks.

3 Basic Formulae and Notations

A *hypercycle* is a subhypergraph $\mathcal{HC}$ induced by the hyperedge set of a cycle $v_1, e_1, v_2, e_2, \ldots, e_m, v_1$. An *elementary* hypercycle is a hypercycle with $d_{\mathcal{HC}}(v_i) = 2$ for $1 \leq i \leq m$ and $d_{\mathcal{HC}}(u) = 1$ for all other $u \in \bigcup_{i=1}^{m} e_i$. When $m \geq 3$, a linear hypercycle is equivalent to being elementary. A hypercycle may be generated by different cycles, but the hyperedges of all of the cycles generating the hypercycle will be the same, up to a permutation.

Acyclic hypergraphs (a *hypertree* when the hypergraph is connected, a *hyperforest* otherwise) and *unicyclic* hypergraphs can now be defined in the natural ways. An acyclic hypergraph contains no hypercycle and a unicyclic hypergraph contains only one hypercycle.

We note that (in [13]) these classes of hypergraphs have been defined by the same structures in an underlying graph spanning the hypergraph, where a graph G *spans* a hypergraph $\mathcal{H}$ if they have the same vertex set and each hyperedge of $\mathcal{H}$ induces a connected subgraph of G. However, there is an inherent ambiguity with this definition, since there are nonisomorphic graphs that span the same hypergraph. We define classes of hypergraphs via hypercycles, which were defined via cycles, or cyclic chains with the condition that $e_i \neq e_j$ for any two hyperedges in the chain, to avoid the ambiguity of spanning graphs. The additional benefit of doing so is the availability of the decomposition theorem with these definitions based on chains. However, the analogous definitions based on spanning graphs used by Kràl et al. [13] provides for larger classes of hypergraphs with interesting properties. We will further this comment in the next section when we define hypercacti.

Chromatic polynomials for many hypergraphs have been expressed in terms of the standard powers of λ [2, 6, 7, 8, 19]. However, chromatic polynomials for cohypergraphs and bihypergraphs can more easily be expressed using terms involving the falling factorial $\lambda^{\underline{k}} = \lambda(\lambda - 1)(\lambda - 2)\ldots(\lambda - k + 1)$. Further, it is also known that when the chromatic polynomial of a mixed hypergraph is expressed in terms of the basis of falling factorial, its coefficients are given by the values of the chromatic spectrum, which count feasible partitions ([18], prop 2.1.1).

In [3], we express some basic formulae in terms of the parameters $\gamma_k(i) = (\lambda - i)^{\underline{k-i}}$ and $\zeta_k(i) = \lambda^{k-i} - \gamma_k(i)$ when $i = 1$ or $i = 2$, and the chromatic polynomials for nonlinear cohypergraphs and bihypergraphs also are easily expressed using the values of these parameters corresponding to higher values of $i \leq k$; we demonstrate this with a result at the conclusion of this paper. Note that the parameter $\gamma_k(i)$ counts the number of rainbows formed using $k - i$ vertices and $\lambda - i$ colors while $\zeta_k(i)$ counts the number of ways to color $k - i$ vertices so that either at least two of the $k - i$ vertices receive the same color from the $\lambda - i$ colors, or

at least one of the $k - i$ vertices receives one of the other i specified colors. It is known (in [3]) and easy to verify that the chromatic polynomials of an isolated hyperedge, cohyperedge and bihyperedge are as follows.

Corollary 3.0.4. *Let e be an isolated hyperedge. Then the chromatic polynomials of e when viewed as a $\mathcal{D}-$hyperedge, $\mathcal{C}-$hyperedge, or $\mathcal{B}-$hyperedge are*

$$
\begin{aligned}
P_\mathcal{D}(e) &= \lambda(\lambda^{|e|-1} - 1) \\
P_\mathcal{C}(e) &= \lambda(\lambda^{|e|-1} - (\lambda - 1)^{\underline{|e|-1}}) = \lambda\zeta_{|e|}(1) \\
P_\mathcal{B}(e) &= \lambda(\lambda^{|e|-1} - (\lambda - 1)^{\underline{|e|-1}} - 1) = \lambda(\zeta_{|e|}(1) - 1)
\end{aligned}
\tag{4}
$$

respectively.

To further shorten the expressions for subsequent chromatic polynomials, we define the *edge type* function $\pi : \mathcal{E} \to \{\mathcal{C}, \mathcal{D}, \mathcal{B}\}$ to specify the coloring condition each hyperedge e is required to follow. Note the edge types can be any nonempty set of equivalence relations on an ordered set. However, in this paper, we are only concerned with those edge types that correspond to the $\mathcal{C}$, $\mathcal{D}$, or $\mathcal{B}$ coloring conditions. Furthermore, we can extend π to any subhypergraph whose hyperedges all have the same edge type. For instance, $\pi(\mathcal{H}_\mathcal{C}) = \mathcal{C}$, $\pi(\mathcal{H}_\mathcal{D}) = \mathcal{D}$, and $\pi(\mathcal{H}_\mathcal{B}) = \mathcal{B}$.

Since each hyperedge of an acyclic linear hypergraph is a block of the hypergraph, we can use corollaries 2.1.3 and 3.0.4 to find the chromatic polynomial of any acyclic linear mixed hypergraph. We state this general formula as

Theorem 3.1. *Let $\mathcal{T}$ be a connected acyclic linear mixed hypergraph with m hyperedges. Then the chromatic polynomial of $\mathcal{T}$ is*

$$
P(\mathcal{T}) = \lambda^{1-m} \prod_{e \in \mathcal{E}} P_{\pi(e)}(e).
\tag{5}
$$

Note, since each factor $P_{\pi(e)}(e)$ itself has λ as a factor, the denominator cancels all but one of the λ factors in the numerator for any $\mathcal{T}$.
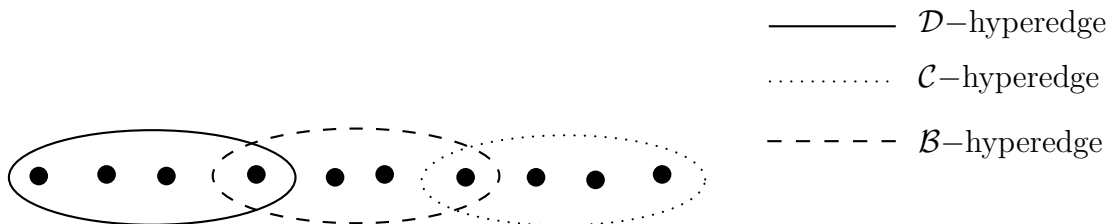

Figure 2: A linear mixed hyperpath

We illustrate Theorem 3.1 with

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Example 3.1. *Let $\mathcal{T}$ be a linear acyclic mixed hypertree with one $5-$hyperedge, one $3-$cohyperedge, and one $4-$bihyperedge. Figure 2 is a representation. Using (4), (5) and Maple 16, we compute the chromatic polynomial in expanded form as*

$$P(\mathcal{T}) = 18\lambda^8 - 45\lambda^7 + 37\lambda^6 - 10\lambda^5 - 18\lambda^4 + 45\lambda^3 - 37\lambda^2 + 10\lambda.$$

Chromatic polynomials for linear uniform mixed hypercycles, though not so simple, have been computed in [3]. We record those results in the following lemma, and defer the proof to section 4. In section 4 we extend the process of [3] to a method of generating the chromatic polynomial of any linear mixed hypercycle, which naturally includes the following lemma as a special case.

Lemma 3.1. *Let $\mathcal{HC}^{k,m}$ be a linear $k-$uniform $m-$hypercycle (with m $k-$hyperedges) with a well-defined edge type and with $m \geq 3$. Then the chromatic polynomials for the three possible edge types are*

$$P_{\mathcal{D}}(\mathcal{HC}^{k,m}) = (\lambda^{k-1} - 1)^m + (-1)^m(\lambda - 1)$$
$$P_{\mathcal{C}}(\mathcal{HC}^{k,m}) = \lambda\left((\gamma_k(2))^{m-2}\left(\lambda^{2k-4} + (\lambda - 1)(\zeta_k(2))^2\right) + \zeta_k(2)\sum_{j=1}^{m-2}(\gamma_k(2))^{j-1}(\zeta_k(1))^{m-j}\right)$$
$$P_{\mathcal{B}}(\mathcal{HC}^{k,m}) = \lambda\Big((\gamma_k(2) - 1)^{m-2}\left(\lambda^{2k-4} + (\lambda - 1)(\zeta_k(2))^2 - 2\lambda^{k-1} + 1\right) +$$
$$\zeta_k(2)\sum_{j=1}^{m-2}(\gamma_k(2) - 1)^{j-1}(\zeta_k(1) - 1)^{m-j}\Big)$$

$$(6)$$

respectively.

It is possible to extend the results in the previous lemma to non-uniform mixed hypercycles as Walter [19] has shown for instance that $P_{\mathcal{D}}(\mathcal{HC}) = \prod_{e \in \mathcal{D}}(\lambda^{|e|} - 1) + (-1)^{|\mathcal{D}|}(\lambda - 1)$.

However, in the interest of this paper, we leave it to the reader as an exercise and discuss in the next section a more general result concerning (non-uniform) mixed hypercycles.

4 Mixed Hypercycles

Let $\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})$ be a linear (elementary) mixed hypercycle with m_1 cohyperedges, m_2 hyperedges, and m_3 bihyperedges.

Lemma 4.1. *The chromatic polynomial of $\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})$ is independent of the order of the hyperedges, justifying the symbol.*

Proof. Let $v_1, v_2, \ldots, v_{m_1+m_2+m_3}$ be the vertices at which two distinct hyperedges of the mixed hypercycle intersect. We can compute the chromatic polynomial of the mixed hypercycle by case work corresponding to counting the number of proper colorings adhering to each possible combination of equality over the colors of each v_i. For instance, when there are three hyperedges there are five cases: $f(v_1) = f(v_2) = f(v_3)$, $f(v_1) = f(v_2) \neq f(v_3)$, $f(v_1) \neq f(v_2) = f(v_3)$, $f(v_1) = f(v_3) \neq f(v_2)$, and $f(v_1), f(v_2), f(v_3)$ are pairwise different.

The contribution to a particular term of the chromatic polynomial made by a single hyperedge is determined by the relationship of the colors at the two endpoint vertices (the two vertices at which the hyperedge intersects other hyperedges). Hence, a permutation of the order of the hyperedges corresponds to a permutation of terms of the chromatic polynomial over the equality relationships of the colors of each v_i. $\square$

We establish some recursive relationships to compute the chromatic polynomial of a mixed hypercycle in terms of mixed hypertrees and mixed hypercycles with one fewer cohyperedge, one fewer bihyperedge, or one fewer hyperedge. As such, we extend our notation so that $\mathcal{T}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})$ is a linear mixed hypertree with m_1 cohyperedges, m_2 hyperedges, and m_3 bihyperedges.

Theorem 4.1. *Let* $\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})$ *be a linear (elementary) mixed hypercycle with* m_1 *cohyperedges,* m_2 *hyperedges, and* m_3 *bihyperedges. Then*
if $m_1 \geq 1$ *and* $m_1 + m_2 + m_3 \geq 3$

$$
\begin{aligned}
&P(\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})) = \\
&\zeta_{|c_{m_1}|}(2)P(\mathcal{T}(c_1, \ldots, c_{m_1-1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})) + \\
&\gamma_{|c_{m_1}|}(2)P(\mathcal{HC}(c_1, \ldots, c_{m_1-1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})),
\end{aligned}
\tag{7}
$$

if $m_3 \geq 1$ *and* $m_1 + m_2 + m_3 \geq 3$

$$
\begin{aligned}
&P(\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})) = \\
&\zeta_{|b_{m_3}|}(2)P(\mathcal{T}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3-1})) + \\
&(\gamma_{|b_{m_3}|}(2) - 1)P(\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3-1})),
\end{aligned}
\tag{8}
$$

and if $m_2 \geq 1$ *and* $m_1 + m_2 + m_3 \geq 3$

$$
\begin{aligned}
&P(\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})) = \\
&\lambda^{|d_{m_2}|-2}P(\mathcal{T}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2-1}; b_1, \ldots, b_{m_3})) - \\
&P(\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2-1}; b_1, \ldots, b_{m_3})).
\end{aligned}
\tag{9}
$$

Proof. (7) Let v_1 and v_2, for convenience, be the endpoints of the cohyperedge c_{m_1}. We can compute the chromatic polynomial of $\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})$ by case work corresponding to a) $f(v_1) = f(v_2)$ and b) $f(v_1) \neq f(v_2)$.

In case a), we contract the hyperedge c_{m_1} by replacing v_1 and v_2 with a single vertex. Since the condition of equality of a pair of colors on c_{m_1} has already been met, the remaining vertices $|c_{m_1}| - 2$ are now isolated and the connected component is chromatically equivalent to $\mathcal{HC}(c_1, \ldots, c_{m_1-1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})$.

In case b) we add a $2-$hyperedge joining v_1 and v_2 in the original hypercycle. The ways of coloring the vertices of $\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3}) - c_{m_1} \cup \{v_1, v_2\}$ ensuring that $f(v_1) \neq f(v_2)$ are given by $P(\mathcal{T}(c_1, \ldots, c_{m_1-1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})) - P(\mathcal{HC}(c_1, \ldots, c_{m_1-1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3}))$, since the first term counts all colorings including when $f(v_1) \neq f(v_2)$ and the second term only counts colorings when $f(v_1) = f(v_2)$. In this case, there are $\lambda^{|c_{m_1}|-2} - (\lambda - 2)^{\underline{|c_{m_1}|-2}}$ ways of coloring the remaining vertices of c_{m_1} avoiding a rainbow. Collecting terms from the two cases now gives (7).

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The proof of (8) is identical to the proof of (7), except in case a) the contracted hypergraph has an additional hyperedge containing the fused vertex replacing v_1 and v_2 and the remaining vertices of b_{m_3}. As such, in this case there are $\lambda^{|b_{m_3}|-2} - 1$ ways of coloring the remaining vertices of b_{m_3} while avoiding a monochrome coloring. The result (8) follows, again after combining terms.

The proof of (9) is identical to (8), except in case b) the remaining $|d_{m_2}| - 2$ vertices of d_{m_2} are isolated.

$\square$

The computation of the chromatic polynomial of any linear mixed hypercycle now reduces to the computation of the chromatic polynomial of linear mixed hypertrees (which are given by Theorem 3.1) and the chromatic polynomials of the following six mixed hypercycles (which can be easily confirmed). For reference, we record them as

Lemma 4.2. *The chromatic polynomial for a mixed hypercycle with two hyperedges is one of the following six polynomials.*

$$P(\mathcal{HC}(d_1, d_2)) = \lambda(\lambda^{|d_1|-2} - 1)(\lambda^{|d_2|-2} - 1) + \lambda(\lambda - 1)(\lambda^{|d_1|-2})(\lambda^{|d_2|-2}) \tag{10}$$

$$P(\mathcal{HC}(c_1, c_2)) = \lambda(\lambda^{|c_1|-2})(\lambda^{|c_2|-2}) + \lambda(\lambda - 1)\zeta_{|c_1|}(2)\zeta_{|c_2|}(2) \tag{11}$$

$$P(\mathcal{HC}(b_1, b_2)) = \lambda(\lambda^{|b_1|-2} - 1)(\lambda^{|b_2|-2} - 1) + \lambda(\lambda - 1)\zeta_{|b_1|}(2)\zeta_{|b_2|}(2) \tag{12}$$

$$P(\mathcal{HC}(c_1; d_1)) = \lambda(\lambda^{|c_1|-2})(\lambda^{|d_1|-2} - 1) + \lambda(\lambda - 1)(\lambda^{|d_1|-2})\zeta_{|c_1|}(2) \tag{13}$$

$$P(\mathcal{HC}(c_1; b_1)) = \lambda(\lambda^{|c_1|-2})(\lambda^{|b_1|-2} - 1) + \lambda(\lambda - 1)\zeta_{|c_1|}(2)\zeta_{|b_1|}(2) \tag{14}$$

$$P(\mathcal{HC}(d_1; b_1)) = \lambda(\lambda^{|d_1|-2} - 1)(\lambda^{|b_1|-2} - 1) + \lambda(\lambda - 1)(\lambda^{|d_1|-2})\zeta_{|b_1|}(2) \tag{15}$$

To illustrate, we find the chromatic polynomial of a non-uniform mixed hypercycle in

Example 4.1. *Let $\mathcal{HC}(c_1, c_2; d_1, d_2; b_1, b_2)$ be a linear mixed hypercycle with the hyperedges in any order and with sizes $|c_1| = 5$, $|c_2| = 6$, $|d_1| = 7$, $|d_2| = 8$, $|b_1| = 3$, $|b_2| = 4$, which can be written as $\mathcal{HC}(5, 6; 7, 8; 3, 4)$. Using Maple 16, we find that the chromatic polynomial is*

$P(\mathcal{HC}(5, 6; 7, 8; 3, 4)) = 2700\lambda^{23} - 32400\lambda^{22} + 184875\lambda^{21} - 647025\lambda^{20} + 1517340\lambda^{19} - 2467905\lambda^{18} + 2796183\lambda^{17} - 2143116\lambda^{16} + 947553\lambda^{15} + 135270\lambda^{14} - 827115\lambda^{13} + 950565\lambda^{12} - 330977\lambda^{11} - 623399\lambda^{10} + 1040846\lambda^9 - 592140\lambda^8 - 341964\lambda^7 + 1392942\lambda^6 - 2252053\lambda^5 + 2397800\lambda^4 - 1663148\lambda^3 + 679008\lambda^2 - 123840\lambda.$

We can compute this polynomial by contracting the hyperedges in any order until we arrive at one of the six basic hypercycles in Lemma 4.2. We leave it as an exercise for the reader to explore other reductions, while we demonstrate what is, perhaps, the easiest order for the reduction. Since recursion (9) is simpler than recursions (7) and (8), we choose to contract $\mathcal{D}-$hyperedges first, when possible. Likewise, we then contract cohyperedges. For Example 4.1, this gives

$$P(\mathcal{HC}(5,6;7,8;3,4)) = \lambda^6 P(\mathcal{T}(5,6;7;3,4)) - \lambda^5 P(\mathcal{T}(5,6;0;3,4)) + \zeta_6(2) P(\mathcal{T}(5;0;3,4)) +$$
$$\gamma_6(2)\zeta_5(2)P(\mathcal{T}(0;0;3,4)) + \gamma_6(2)\gamma_5(2)P(\mathcal{HC}(0;0;3,4)).$$

It is an easy, though a very laborious algebraic exercise to compute the chromatic polynomials of mixed hypercycles recursively. We choose to demonstrate the general form used in Example 4.1 when $m_3 \geq 2$ and leave other reductions to the reader.

Corollary 4.1.1. *Let* $\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})$ *be a linear (elementary) hypercycle with* m_1 *cohyperedges,* m_2 *hyperedges, and* m_3 *bihyperedges with* $m_3 \geq 2$. *Then*
$$P(\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2}; b_1, \ldots, b_{m_3})) =$$
$$\lambda^{|d_{m_2}|-2} P(\mathcal{T}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2-1}; b_1, \ldots, b_{m_3})) -$$
$$\lambda^{|d_{m_2-1}|-2} P(\mathcal{HC}(c_1, \ldots, c_{m_1}; d_1, \ldots, d_{m_2-2}; b_1, \ldots, b_{m_3})) + \ldots +$$
$$(-1)^{m_2-1}\lambda^{|d_1|-2} P(\mathcal{T}(c_1, \ldots, c_{m_1}; 0; b_1, \ldots, b_{m_3})) +$$
$$(-1)^{m_2} P(\mathcal{HC}(c_1, \ldots, c_{m_1}; 0; b_1, \ldots, b_{m_3}))$$

where

$$P(\mathcal{HC}(c_1, \ldots, c_{m_1}; 0; b_1, \ldots, b_{m_3})) = \zeta_{|c_{m_1}|}(2) P(\mathcal{T}(c_1, \ldots, c_{m_1-1}; 0; b_1, \ldots, b_{m_3})) +$$
$$\gamma_{|c_{m_1}|}(2)\zeta_{|c_{m_1}-1|}(2) P(\mathcal{T}(c_1, \ldots, c_{m_1-2}; 0; b_1, \ldots, b_{m_3})) + \ldots +$$
$$\gamma_{|c_{m_1}|}(2) \ldots \gamma_{|c_2|}(2)\zeta_{|c_1|}(2) P(\mathcal{T}(0; 0; b_1, \ldots, b_{m_3})) +$$
$$\gamma_{|c_{m_1}|}(2) \ldots \gamma_{|c_2|}(2)\gamma_{|c_1|}(2) P(\mathcal{HC}(0; 0; b_1, \ldots, b_{m_3}))$$

where

$$P(\mathcal{HC}(0; 0; b_1, \ldots, b_{m_3})) = \zeta_{|b_{m_3}|}(2) P(\mathcal{T}(0; 0; b_1, \ldots, b_{m_3-1})) +$$
$$(\gamma_{|b_{m_3}|}(2) - 1)\zeta_{|b_{m_3}-1|}(2) P(\mathcal{T}(0; 0; b_1, \ldots, b_{m_3-2})) + \ldots +$$
$$(\gamma_{|b_{m_3}|}(2) - 1) \ldots (\gamma_{|b_4|}(2) - 1)\zeta_{|b_3|}(2) P(\mathcal{T}(0; 0; b_1, b_2)) +$$
$$(\gamma_{|b_{m_3}|}(2) - 1) \ldots (\gamma_{|b_3|}(2) - 1)\Big(\lambda(\lambda^{|b_2|-2} - 1)(\lambda^{|b_1|-2} - 1) + \lambda(\lambda - 1)\zeta_{|b_2|}(2)\zeta_{|b_1|}(2)\Big). \tag{16}$$

When the hypercycle is $k-$uniform and of a well-defined edge type, we can simplify the recursive expression and substitute the corresponding formulae for mixed hypertrees to obtain (6). For instance, the third expression of (6) is just (16) when all of the sizes of the bihyperedges are k.

5   Mixed Hypercacti

Hypercacti are hypergraphs whose blocks are either elementary hypercycles or acyclic subhypergraphs. In this section we work with linear hypercacti, where blocks are either elementary

hypercycles or hyperedges.

As a side note on these definitions, our class of hypercacti is different from those studied by Kràl et al.[13]; because, a cactus graph can span a hypergraph that does not decompose into blocks such as the examples used by Kràl et al. to show that some (weak) hypercacti have broken feasible sets (Theorem 3, [13]). If we remove the restriction in our definition that blocks of hypercacti are elementary hypercycles and allow for arbitrary hypercyclic blocks, then we allow for the complexity exhibited in examples such as those cited from Kràl et al. Restricting ourselves to linear mixed hypergraphs, the confusion over these terms does not affect us in this paper.

We can now find the chromatic polynomials for any linear mixed hypercactus. The general expression immediately follows from Corollaries 2.1.3 and 3.0.4 and Lemma 3.1, and we record it as

Theorem 5.1. *Let $\mathcal{H}$ be a connected linear mixed hypercactus with blocks arranged into subhypergraphs $\mathcal{T}_1, \ldots, \mathcal{T}_{n_1}$, $\mathcal{HC}_1, \ldots, \mathcal{HC}_{n_2}$ where each $\mathcal{T}_i$ is a connected acyclic linear mixed hypergraph and each block $\mathcal{HC}_j$ is a linear mixed hypercycle. Then the chromatic polynomial of $\mathcal{H}$ is*

$$P(\mathcal{H}) = \lambda^{1-n_1-n_2} \prod_{i=1}^{n_1} P(\mathcal{T}_i) \prod_{j=1}^{n_2} P(\mathcal{HC}_j). \tag{17}$$
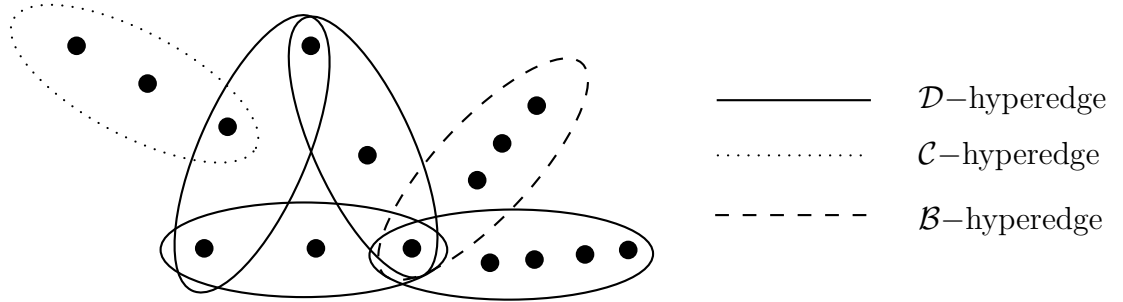

Figure 3: A unicyclic mixed hypercycle

As a special case, where the hypercyclic blocks are uniform and of a well-defined edge type, we record the adapted formula as a corollary, since this case uses the reduced formulae recorded in (6) and provides examples which are more easily computed.

Corollary 5.1.1. *Let $\mathcal{H}$ be a connected linear mixed hypercactus with blocks arranged into subhypergraphs $\mathcal{T}_1, \ldots, \mathcal{T}_{n_1}$, $\mathcal{HC}_1^{k_1, m_1}, \ldots, \mathcal{HC}_{n_2}^{k_{n_2}, m_{n_2}}$ where each $\mathcal{T}_i$ is a connected acyclic linear mixed hypergraph and each block $\mathcal{HC}_j^{k_j, m_j}$ is a linear mixed k_j-uniform m_j-hypercycle with a well-defined edge type and with $m_j \geq 3$. Then the chromatic polynomial of $\mathcal{H}$ is*

$$P(\mathcal{H}) = \lambda^{1-n_1-n_2} \prod_{i=1}^{n_1} P(\mathcal{T}_i) \prod_{j=1}^{n_2} P_{\pi(\mathcal{HC}_j^{k_j, m_j})}(\mathcal{HC}_j^{k_j, m_j}). \tag{18}$$

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Note: Since each factor of the numerator corresponding to one of the listed subhypergraphs has a factor of λ with multiplicity 1, by examination of (6) and the comment following Theorem 3.1, the denominator of (17) will be canceled, leaving a factor of λ with multiplicity 1 in a reduced form of the polynomial. We now look at several examples to illustrate Corollary 5.1.1.

Example 5.1. *Let $\mathcal{H}_1$ be a linear unicyclic mixed hypergraph with one $3-$uniform $3-$hypercycle, one $5-$hyperedge, one $3-$cohyperedge, and one $4-$bihyperdege. Figure 3 shows this mixed hypergraph as a "hairy mixed hypercycle" ([13]), but other non-isomorphic arrangements will have the same chromatic polynomial.*

Using (4),(5),(6),(18) and Maple 16, we compute the chromatic polynomial in expanded form as

$P(\mathcal{H}_1) = 18\lambda^{13} - 45\lambda^{12} - 17\lambda^{11} + 125\lambda^{10} - 75\lambda^9 - 78\lambda^8 + 173\lambda^7 - 192\lambda^6 + 67\lambda^5 + 123\lambda^4 - 156\lambda^3 + 67\lambda^2 - 10\lambda.$

Example 5.2. *Let $\mathcal{H}_2$ be a linear mixed hypercactus with one $3-$uniform hypercycle of each edge type, one $5-$cohyperedge, and one $4-$hyperedge. Figure 4 shows a version of this mixed hypercactus arranged in such a way that the hyperedges separate the cycles. Kràl et al. use the term weak mixed hypercactus for such a mixed hypergraph and show that some weak mixed hypercacti have gaps in their feasible domain. Using (4),(5),(6),(18) and Maple 16, we compute the chromatic polynomial in expanded form.*
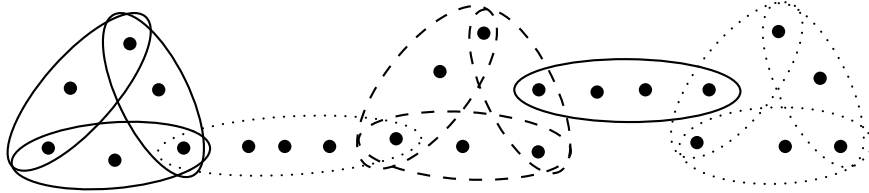

Figure 4: A weak mixed hypercactus

$P(\mathcal{H}_2) = 10\lambda^{18} + 335\lambda^{17} + 1315\lambda^{16} - 22944\lambda^{15} + 83457\lambda^{14} - 116239\lambda^{13} - 16361\lambda^{12} + 236106\lambda^{11} - 198736\lambda^{10} - 129724\lambda^9 + 336372\lambda^8 - 349896\lambda^7 + 513731\lambda^6 - 749582\lambda^5 + 673924\lambda^4 - 344712\lambda^3 + 93312\lambda^2 - 10368\lambda.$

Example 5.3. *Let $\mathcal{H}_3$ be a linear mixed hypercactus with one $3-$uniform bihypercycle, one $4-$uniform hypercycle, and one $5-$uniform cohypercycle. Figure 5 shows a version of this hypergraph. Kràl et al.[13] call such mixed hypercacti, strong, and show that the feasible set of any such mixed hypergraph is gap free. Using (4),(6),(17) and Maple 16, we compute the chromatic polynomial in expanded form as*

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$$P(\mathcal{H}_3) = \lambda^{21} + 989\lambda^{20} + 6327\lambda^{19} - 167076\lambda^{18} + 1173697\lambda^{17} - 4671361\lambda^{16} + 12552218\lambda^{15} - 25154549\lambda^{14} + 41274344\lambda^{13} - 60558188\lambda^{12} + 82266525\lambda^{11} - 100474755\lambda^{10} + 107431796\lambda^{9} - 101811056\lambda^{8} + 84213724\lambda^{7} - 52320940\lambda^{6} + 14282640\lambda^{5} + 9532944\lambda^{4} - 11620800\lambda^{3} + 4790016\lambda^{2} - 746496\lambda.$$

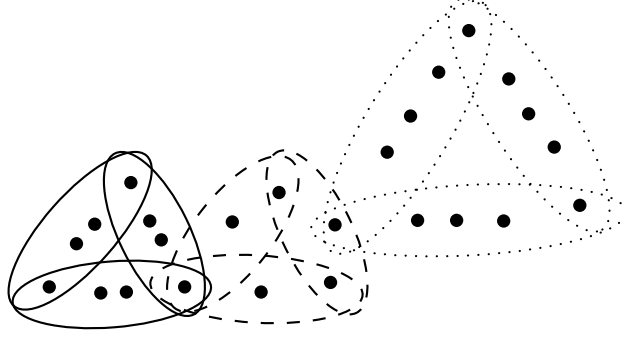

Figure 5: A strong mixed hypercactus

6 Mixed $\Theta-$hypergraphs

Thus far, we have only considered mixed hypergraphs whose chromatic polynomials can be decomposed as in Corollary 2.1.3 and factors are given by the basic formulae in (4), and (6). It is immediately apparent that generalizations are possible by either extending the dictionary of the basic formulae or by considering more general separators. As a step in this direction, we define a class of multi-bridge mixed hypergraphs that are an extension of $\Theta-$graphs [10], which we call *mixed $\Theta-$hypergraphs*. See [6, 7, 10] for other work with the hypergraphs of this class.

Let Θ_r be a connected mixed hypergraph with separator $\overline{K_2} \cong \Theta_r[\{u, v\}]$, the empty graph on two vertices u and v, whose derived subhypergraphs $\mathcal{T}_1, \ldots, \mathcal{T}_r$ are acyclic mixed hypergraphs. Clearly each $\mathcal{T}_i$ is a bridge between the vertices u and v.

Since $\overline{K_2}$ is not uniquely colorable, we cannot immediately decompose the chromatic polynomial of a mixed $\Theta-$hypergraph using Corollary 2.1.1. However, a fundamental reduction of any chromatic polynomial reduces the computations to the two cases when 1) $f(u) = f(v)$ and 2) $f(u) \neq f(v)$ for a specified pair of vertices. Case 1) corresponds to identifying the two vertices u and v by a single vertex (the result of contracting an added $2-$hyperedge), and case 2) corresponds to connecting u and v by an additional $2-$hyperedge.

In case 1) the new hypergraph is simply a (strong) mixed hypercactus with mixed hypercycles $\mathcal{HC}_i$ separated by the new vertex. As such, the first component of the chromatic polynomial of Θ_r is given by (17), without the mixed hypertree factors.

In case 2) the new hypergraph Θ_r^* is separated by the complete graph on two vertices K_2 and its chromatic polynomial can be found using (2) in Corollary 2.1.1. The derived subhypergraphs of Θ_r^* are $\mathcal{T}_1^*, \ldots, \mathcal{T}_r^*$ where each $\mathcal{T}_i^*$ is $\mathcal{T}_i$ with the additional $2-$hyperedge

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connecting u and v. Observing that $P(\mathcal{T}_i^*) = P(\mathcal{T}_i) - P(\mathcal{HC}_i)$, since $P(\mathcal{HC}_i)$ counts the proper colorings of $\mathcal{T}_i$ when $f(u) = f(v)$ and $P(\mathcal{T}_i^*)$ only counts the colorings of $\mathcal{T}_i$ when $f(u) \neq f(v)$, completes the proof of

Theorem 6.1. *Let Θ_r be a connected mixed hypergraph with separator $\overline{K_2} \cong \Theta_r[\{u, v\}]$, the empty graph on two vertices u and v, whose derived subhypergraphs $\mathcal{T}_1, \ldots, \mathcal{T}_r$ are acyclic mixed hypergraphs. Furthermore, let $\mathcal{HC}_i$ be the mixed hypercycle formed by identifying the two vertices u and v into a single vertex in $\mathcal{T}_i$. Then*

$$P(\Theta_r) = \lambda^{1-r} \prod_{i=1}^{r} P(\mathcal{HC}_i) + \lambda^{1-r}(1 - \lambda)^{1-r} \prod_{i=1}^{r} \Big(P(\mathcal{T}_i) - P(\mathcal{HC}_i) \Big). \qquad (19)$$

We leave it as an exercise for the reader to compute the chromatic polynomials of mixed Θ−hypergraphs with non-uniform mixed bridges (using Theorem 3.1 and the remark following Lemma 3.1). In interest of space we illustrate the process, using expressions (6), to find an explicit expression for the chromatic polynomial of Θ_r when each $\mathcal{T}_i$ bridge is a k_i−uniform linear mixed hypertree with m_i hyperedges of the same edge type, so that each $\mathcal{HC}_i$ is a hypercycle $\mathcal{HC}^{k_i,m_i}$ with well-defined edge type.

Example 6.1. *Let Θ_3 be the mixed Θ−hypergraph with $\mathcal{T}_1$ a 5−uniform linear hyperpath of length 4, $\mathcal{T}_2$ a 3−uniform linear bihyperpath of length 3, and $\mathcal{T}_3$ a 4−uniform linear cohyperpath of length 5. Figure 6 is a representation.*
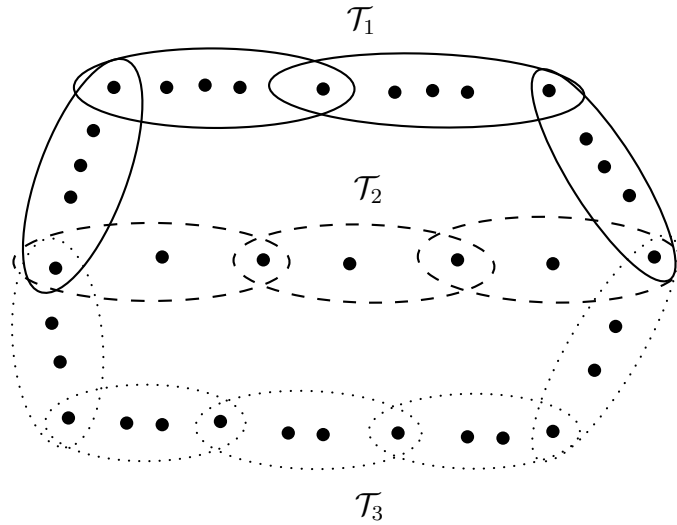

Figure 6: A mixed Θ−hypergraph

Then, using the notations in expression (4) and (6), and following (5) and (19),

we have $P(\Theta_3) = \lambda^{-2} P_{\mathcal{D}}(\mathcal{HC}^{5,4}) P_{\mathcal{B}}(\mathcal{HC}^{3,3}) P_{\mathcal{C}}(\mathcal{HC}^{4,5}) + \lambda^{-2}(\lambda - 1)^{-2}\left(\lambda^{-3}\left(P_{\mathcal{D}}(e_1)\right)^4 - \right.$

$P_{\mathcal{D}}(\mathcal{HC}^{5,4})\Big)\left(\lambda^{-2}\left(P_{\mathcal{B}}(e_2)\right)^3 - P_{\mathcal{B}}(\mathcal{HC}^{3,3})\right)\left(\lambda^{-4}\left(P_{\mathcal{C}}(e_3)\right)^5 - P_{\mathcal{C}}(\mathcal{HC}^{4,5})\right)$ *where e_i is any hy-peredge of $\mathcal{T}_i$. Using Maple 16 to expand, we find*

$P(\Theta_3) = \lambda^{29} + 209917\lambda^{28} - 2553850\lambda^{27} + 14504412\lambda^{26} - 50900694\lambda^{25} + 122254132\lambda^{24} - 206276727\lambda^{23} + 226570863\lambda^{22} - 78922300\lambda^{21} - 280202606\lambda^{20} + 734764864\lambda^{19} - 1006697220\lambda^{18} + 813848600\lambda^{17} - 104669524\lambda^{16} - 825215186\lambda^{15} + 1470992011\lambda^{14} - 1448149789\lambda^{13} + 768125234\lambda^{12} + 169428056\lambda^{11} - 862855382\lambda^{10} + 1038127864\lambda^{9} - 77398819\lambda^{8} + 362123049\lambda^{7} - 58306248\lambda^{6} - 60471792\lambda^{5} + 59166288\lambda^{4} - 27064368\lambda^{3} + 6998400\lambda^{2} - 839808\lambda.$

7   Comments

To complete the comment we started at the beginning of section 6, we close this paper with the formulae for another class of mixed hypergraphs. These hypergraphs appear as blocks of a super hypergraph, along with mixed acyclic and hypercyclic blocks, and Corollary 2.1.3 says the chromatic polynomial splits with factors given by these subhypergraphs. The decomposition via case work in the proof of Theorem 6.1 also illustrates a brute force way to decompose the chromatic polynomials of mixed hypergraphs which are separated by non-uniquely colorable subhypergraphs. Such a process certainly grows in complexity with non-polynomial growth. Thus, the formulae for mixed hypergraphs which separators are not uniquely colorable are often obtained through a combinatorial argument, when it is possible. As an example, we present here the result of a nonlinear mixed hypergraph after its definition [4].

A *sunflower (hypergraph)* $\mathcal{S} = (X, \mathcal{E})$ with l *petals* and a *core* S is a collection of sets $e_1, \ldots, e_l$ such that $e_i \cap e_j = S$ for all $i \neq j$. The elements of the core are called *seeds*. Figure 1 is an example of a $3-$uniform sunflower with 2 seeds. When all the petals of $\mathcal{S}$ are of type $\mathcal{D}$, $\mathcal{C}$ or $\mathcal{B}$, the resulting mixed hypergraph is called a $\mathcal{D}-$, $\mathcal{C}-$ or $\mathcal{B}-$sunflower, respectively.

The formula for the chromatic polynomial of a non-uniform $\mathcal{D}-$sunflower was first obtained by Walter [19]. We report his result here as a corollary (with a slightly different notation) after we present an extension of that result to mixed sunflowers with petals of both types $\mathcal{D}$ and $\mathcal{C}$. We note that when $|S| = 1$, a mixed sunflower is isomorphic to a mixed hyperstar which chromatic polynomial is given by Theorem 3.1. Thus, for the next results, we assume $|S| \geq 2$ and require that $|e| - |S| > 0$ for each $e \in \mathcal{E} = \mathcal{C} \cup \mathcal{D}$.

Theorem 7.1. *Let $\mathcal{S} = (X, \mathcal{C}, \mathcal{D})$ be any mixed sunflower. Then*

$$P(\mathcal{S}, \lambda) = \lambda \prod_{\substack{d \in \mathcal{D} \\ c \in \mathcal{C}}} \lambda^{|c|-|S|}(\lambda^{|d|-|S|} - 1) + \lambda^{\underline{|S|}} \prod_{\substack{d \in \mathcal{D} \\ c \in \mathcal{C}}} \lambda^{|d|-|S|} \zeta_{|c|}(|S|) + \lambda(\zeta_{|S|}(1) - 1) \prod_{\substack{d \in \mathcal{D} \\ c \in \mathcal{C}}} \lambda^{|d|-|S|} \lambda^{|c|-|S|}.$$

Proof. In any proper coloring of the core S, either one of the following is true:

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(i) S is rainbow (or polychromatic). Then there are $\lambda^{|S|}$ ways to color its seeds. For each such coloring, there are $\lambda^{|c|-|S|} - (\lambda - |S|)^{\underline{|c|-|S|}} = \zeta_{|c|}(|S|)$ and $\lambda^{|d|-|S|}$ ways to color (independently) the remaining vertices of each petal $c \in \mathcal{C}$ and each petal $d \in \mathcal{D}$, respectively. This ensures the condition that no $\mathcal{C}-$petal is rainbow and no $\mathcal{D}-$petal is monochromatic, giving the middle term.

(ii) S is monochromatic. In this case, there are $\lambda^{|c|-|S|}$ and $\lambda^{|d|-|S|} - 1$ ways to color the remaining vertices of each $\mathcal{C}-$petal and each $\mathcal{D}-$petal, respectively. Thus, the first term gives the number of proper colorings.

(iii) S is neither polychromatic nor monochromatic. There are $\lambda^{|S|} - \lambda^{\underline{|S|}} - \lambda = \lambda(\zeta_{|S|}(1) - 1)$ ways to color its seeds. The number of proper colorings in this case is therefore given by the last term.

□

Corollaries 7.1.1 and 7.1.2 follow from Theorem 7.1 when $\mathcal{D} = \emptyset$ and $\mathcal{C} = \emptyset$, respectively (after expansion of the last term).

Corollary 7.1.1. *Let $\mathcal{S} = (X, \mathcal{C})$ be any $\mathcal{C}-$sunflower. Then*

$$P(\mathcal{S}, \lambda) = \lambda^n - \lambda^{\underline{|S|}} \prod_{c \in \mathcal{C}} \lambda^{|c|-|S|} + \lambda^{\underline{|S|}} \prod_{c \in \mathcal{C}} \zeta_{|c|}(|S|).$$

Corollary 7.1.2. *Let $\mathcal{S} = (X, \mathcal{D})$ be any $\mathcal{D}-$sunflower. Then*

$$P(\mathcal{S}, \lambda) = \lambda^n - \lambda^{n-|S|+1} + \lambda \prod_{d \in \mathcal{D}} (\lambda^{|d|-|S|} - 1).$$

We point out that Walter's original proof of Corollary 7.1.2 is by induction ([19], Theorem 2.2).

Work such as this to compute the chromatic polynomials of some mixed hypergraphs are useful in some specific applications (see [18]), and we hope aids in experimentation toward the goal of interpreting the coefficients of these polynomials, which remains an open problem.

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